Simple upper bound on the information rate of the phase noise channel

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A simple upper bound on the information rate (IR) of digital communications over phase noise-impaired channels is proposed. In particular, the transmission of linearly modulated signals, such as QAM, at high-signal-to-noise ratio is considered, thus extending previous literature results for phase modulated signals, e.g. PSK. A simulation-based lower bound on the IR shows that the proposed bound is tight for small-medium size constellations and large phase noise.

Introduction: Phase noise collectively denotes unwanted random fluctuations in the phase of the waveform generated by an oscillator and can cause distortion and performance degradation in digital communication systems [1]. Phase noise-impaired channels are of interest in several scenarios, including millimetre-wave digital wireless communications [2], distributed MIMO systems [3], and optical communications [4, 5].

Phase noise-limited communications can be analysed from various viewpoints. In particular, the achievable information rate (IR), i.e. the maximum amount of information bits which can be reliably transmitted over the considered channel with a given modulation format, has been considered as it can give insights on the design of practical communication systems, see, e.g. [4, 6] and references therein. In [6], the author presents a simple closed-form upper bound for phase shift keying (PSK) modulations derived in the high-signal-to-noise ratio (SNR) regime, but valid for any SNR.

In this Letter, we extend the bound of [6] to amplitude and phase modulations, e.g. quadrature amplitude modulation (QAM) or amplitude and PSK (APSK). Simulation-based estimation of a lower bound on the IR shows that the proposed upper bound is tight. This upper bound can be useful to predict the performance of linear modulations in the high-SNR regime, where the simulation-based lower bound may exhibit numerical problems [4].

System model: We consider the transmission of linearly modulated data symbols $\{x_k\}$, independent and uniformly distributed onto a constellation of cardinality M. Assuming Nyquist shaping with matched filtering and slowly variant phase noise, the sampled sequence at the channel output is [6]

$$y_k = x_k \, \mathrm{e}^{\mathrm{j}\theta_k} + n_k$$

where $\{n_k\}$ are independent and identically distributed (i.i.d.) additive white Gaussian noise samples. The phase noise process can be described as first-order Markov, where

$$\theta_k = \theta_{k-1} + \Delta_k \quad \mod 2\pi \tag{1}$$

with i.i.d. increments $\Delta_k \sim \mathcal{N}(0, \sigma_\Delta^2)$. Therefore, the conditional probability density function (PDF), given the previous realisation, depends only on the phase difference and has the following wrapped Gaussian form on $[0, 2\pi)$ [6, 7]

$$p(\theta_k|\theta_{k-1}) = \frac{1}{\sqrt{2\pi\sigma_\Delta^2}} \sum_{\ell=-\infty}^{\infty} \mathrm{e}^{-\left((\theta_k - 2\pi\ell - \theta_{k-1})^2/2\sigma_\Delta^2\right)}$$

Using a polar decomposition, the generic kth transmitted and received symbols can be expressed as

$$x_k = a_k \, \mathrm{e}^{\mathrm{j}\varphi_k} \tag{2}$$

$$y_k = b_k \, \mathrm{e}^{\mathrm{j}\psi_k} \tag{3}$$

where a_k and b_k are the input and output amplitudes, respectively, whereas φ_k and ψ_k are the input and output phases, respectively. The transmitted symbol amplitude a_k is a random variable which belongs to a set of discretised radii $\{r_1, r_2, \ldots, r_L\}$, being L the number of possible amplitude values, each of them with probability

$$p_i = \frac{\#\,\text{points with radius } r_i}{M} \qquad i = 1, 2, \ldots, L$$

The values r_i and p_i, $i = 1, 2, \ldots, L$, depend on the considered constellation.

A pictorial description for 16-QAM constellation, where $L = 3$, is shown in Fig. 1.

In this case, denoting by d the distance between adjacent points, it can be easily shown that $r_1 = d/\sqrt{2}$, $r_2 = d\sqrt{10}/2$, and $r_3 = 3d/\sqrt{2}$ with probability $p_1 = 1/4$, $p_2 = 1/2$, and $p_3 = 1/4$, respectively. Note the sub-constellations described by the inner and outer circles are standard PSK constellations, whereas the intermediate one (i.e. that with radius r_2) is not, due to the non-equally spaced points on the circle.

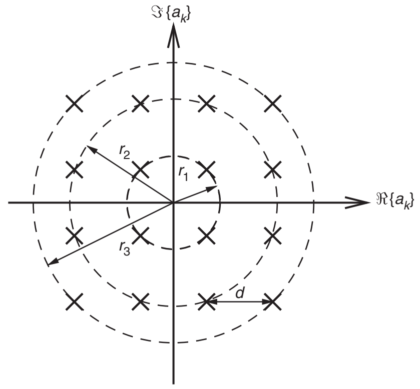

Fig. 1 *Polar decomposition for 16-QAM constellation*

Simple upper bound: We now extend the upper bound, described in [6] for phase modulations, to general linear modulations. Since the IR is a non-decreasing function of the SNR, the bound will be derived assuming a high-SNR regime, where the observation model is

$$y_k = x_k \, \mathrm{e}^{\mathrm{j}\theta_k} \tag{4}$$

so that, according to (2) and (3), $b_k = a_k$ and $\psi_k = \varphi_k + \theta_k$.

Given the channel input and output stochastic processes $\mathcal{X}$ and $\mathcal{Y}$, respectively, the IR is defined as

$$I(\mathcal{X}; \mathcal{Y}) = \lim_{N \to +\infty} \frac{1}{N} I(x_1^N; y_1^N)$$

where N is the length of the information sequence and the notation z_a^b $(a < b)$ stands for the sequence $\{z_a, \ldots, z_b\}$. Following the approach in [5], by resorting to a proper application of the chain rule the IR can be decomposed as

$$I(\mathcal{X}; \mathcal{Y}) = I(\mathcal{A}; \mathcal{B}) + I(\Phi; \Psi|\mathcal{A}) + I(\mathcal{A}; \Psi|\mathcal{B}) + I(\Phi; \mathcal{B}|\mathcal{A}, \Psi)$$

where $\mathcal{A}$ and $\mathcal{B}$ are the stochastic processes associated with input and output amplitudes, respectively, whereas Φ and Ψ are the stochastic processes associated with input and output phases, respectively.

In the high-SNR regime, the third and fourth term, corresponding to mixed amplitude/phase terms, are null. In fact, the third term

$$I(\mathcal{A}; \Psi|\mathcal{B}) = H(\mathcal{A}|\mathcal{B}) - H(\mathcal{A}|\Psi, \mathcal{B}) = 0$$

where $H(\cdot)$ is the discrete entropy rate, since, at high SNR, $\mathcal{A}$ is perfectly known given $\mathcal{B}$, i.e. $H(\mathcal{A}|\mathcal{B}) = H(\mathcal{A}|\Psi, \mathcal{B}) = 0$. The fourth term

$$I(\Phi; \mathcal{B}|\mathcal{A}, \Psi) = H(\Phi|\mathcal{A}, \Psi) - H(\Phi|\mathcal{A}, \mathcal{B}, \Psi) = 0$$

because $H(\Phi|\mathcal{A}, \Psi) = H(\Phi|\mathcal{A}, \mathcal{B}, \Psi)$, as $\mathcal{B}$ is perfectly known given $\mathcal{A}$. Hence, the IR at any value of SNR can be upper bounded as

$$I(\mathcal{X}; \mathcal{Y}) \le I(\mathcal{A}; \mathcal{B}) + I(\Phi; \Psi|\mathcal{A}) \tag{5}$$

The amplitude term in (5) can be expressed as

$$I(\mathcal{A}; \mathcal{B}) = H(\mathcal{A}) - H(\mathcal{A}|\mathcal{B}) = H(\mathcal{A})$$

where $H(\mathcal{A}|\mathcal{B}) = 0$ due to (4). Therefore, one can write

$$\begin{aligned} I(\mathcal{X}; \mathcal{Y}) &\le I(\mathcal{A}; \mathcal{B}) + I(\Phi; \Psi|\mathcal{A}) \\ &= H(\mathcal{A}) + h(\Psi|\mathcal{A}) - h(\Psi|\Phi, \mathcal{A}) \end{aligned} \tag{6}$$

where $h(\cdot)$ denotes the differential entropy rate of a continuous stochastic process. If $\sigma_\Delta \gg 1$, the received phase is uniformly distributed on $[0, 2\pi)$ so that $h(\Psi|\mathcal{A}) \simeq h(\Psi|\Phi, \mathcal{A})$. Therefore, (6) reduces to

$$I(\mathcal{X}; \mathcal{Y}) \le H(\mathcal{A}) \qquad \sigma_\Delta \gg 1 \tag{7}$$

i.e. for large phase noise intensity, only the information associated with the amplitude component can be transmitted reliably over the channel.

The differential entropy $h(\Psi|\mathcal{A})$ in (6) can be upper bounded by that of a random variable uniformly distributed on $[0, 2\pi)$. The entropy rate $h(\Psi|\Phi, \mathcal{A})$ can be evaluated by resorting to the definition of entropy rate of a stochastic process and the chain rule

$$h(\Psi|\Phi, \mathcal{A}) = \lim_{N\to+\infty} \frac{1}{N} h(\psi_1^N|\varphi_1^N, a_1^N)$$
$$= \lim_{N\to+\infty} \frac{1}{N}\left[h(\psi_1|\varphi_1, a_1) + \sum_{k=2}^{N} h(\psi_k|\psi_{k-1}, \varphi_{k-1}^k, a_{k-1}^k)\right]$$

By definition of the Wiener phase noise process, $h(\psi_1|\varphi_1, a_1) = h(\theta_1) = \log_2(2\pi)$ bit, since the wrapped phase channel rotation has marginally a uniform distribution on $[0, 2\pi)$, and

$$h(\psi_k|\psi_{k-1}, \varphi_{k-1}^k, a_{k-1}^k) = h(\Delta) \quad \forall k \tag{8}$$

where Δ is the stochastic process of i.i.d. wrapped Gaussian variables. In fact, the output phase at time instant k, for fixed output phase at time instant $k-1$ and input phases at time instants k and $k-1$, only depends on the Gaussian increment Δ_k. The differential entropy rate $h(\Delta)$ in bits can be found in [8] as

$$h(\Delta) = \log_2(2\pi) - \sum_{i=1}^{+\infty} \log_2(1-q^i) + \frac{2}{\ln 2}\sum_{i=1}^{+\infty} \frac{(-1)^i}{i}\frac{q^{(i^2+i)/2}}{1-q^i}$$

where $q = \mathrm{e}^{-\sigma_\Delta^2}$. Using (8) into (6), one has

$$I(\mathcal{X}; \mathcal{Y}) \le H(\mathcal{A}) + \log_2(2\pi) - h(\Delta) \quad \text{bit/channel use}$$

Finally, since for a fixed constellation size M the IR cannot be greater than $\log_2 M$ bit/channel use (in particular for small phase noise intensities), the desired upper bound can be expressed as

$$\bar{I}(\mathcal{X}; \mathcal{Y}) = \min\{\log_2 M, H(\mathcal{A}) + \log_2(2\pi) - h(\Delta)\}$$

Numerical results: In this section, we provide numerical results for the presented upper bound and compare it with a simulation-based estimation of a lower bound in order to investigate its tightness. Simulation results are obtained by the method discussed in [9], which is now briefly recalled. The IR of a given channel can be estimated as

$$I(\mathcal{X}; \mathcal{Y}) \simeq -\frac{1}{N}\log p(y_1^N) + \frac{1}{N}\log p(y_1^N|x_1^N) \tag{9}$$

where the approximation holds for sufficiently large N. Since the PDFs required by (9) may not be available in closed-form, we can exploit the *auxiliary channel theorem* that allows to estimate upper and lower bounds on $I(\mathcal{X}; \mathcal{Y})$ [9]. If $q(y_1^N|x_1^N)$ is the PDF that describes an auxiliary channel which approximates the 'true' one and $q(y_1^N) = \sum_{x_1^N\in\mathcal{X}^n} P(x_1^N)q(y_1^N|x_1^N)$ is the PDF of the output of this auxiliary channel, the following limit:

$$\underline{I}(\mathcal{X}; \mathcal{Y}) = \lim_{N\to+\infty} -\frac{1}{N}\log q(y_1^N) + \frac{1}{N}\log q(y_1^N|x_1^N) \tag{10}$$

is a lower bound on the desired IR, i.e. $\underline{I}(\mathcal{X}; \mathcal{Y}) \le I(\mathcal{X}; \mathcal{Y})$.

This bound can be estimated by approximating the limit (10) with a value for sufficiently large N and using the forward recursion of the Bahl, Cocke, Jelinek, Raviv (BCJR) algorithm to evaluate the required quantities [9]. As already done in [4–7], the trellis state for the BCJR is defined as the quantised phase values

$$\phi_i = \frac{2\pi}{\mathcal{S}} i \quad i = 0, 1, \ldots, \mathcal{S}-1$$

where $\mathcal{S}$ is the number of quantisation levels. The transition probabilities between states depend on σ_Δ according to the following expression [7]

$$P(\phi_m|\phi_j) = \frac{\mathcal{S}}{2\pi}\sum_{\ell=-\infty}^{\infty}\int_{(\pi/\mathcal{S})(2j-1)}^{(\pi/\mathcal{S})(2j+1)} \chi_{m\ell}(\rho)\mathrm{d}\rho \tag{11}$$

where

$$\chi_{m\ell}(\rho) = Q\left(\frac{(\pi/\mathcal{S})(2m+1-2\ell\mathcal{S})-\rho}{\sigma_\delta}\right) - Q\left(\frac{(\pi/\mathcal{S})(2m-1-2\ell\mathcal{S})-\rho}{\sigma_\delta}\right)$$

in which $Q(\cdot)$ is the Gaussian tail function.

In the following, we shall consider $\mathcal{S} = 256$ for small constellations, up to 16-QAM, whereas $\mathcal{S} = 512$ for medium-large constellation sizes, from 64-QAM on.

In Fig. 2, the IR is shown, as a function of σ_Δ, for various QAM modulation schemes. The presented upper bound (solid lines) is compared with simulation results (dashed lines) relative to the estimated lower bound (10) at high SNR.

Note that, as predicted by (7), for large values of σ_Δ the upper bound approaches the value $H(\mathcal{A})$. Moreover, the proposed bound is tight for small-medium constellation sizes, e.g. up to 64-QAM, for all considered values of σ_Δ. For larger constellation sizes, e.g. 256- or 1024-QAM, the difference between the upper and lower bounds is more evident for two reasons. First, the simulation-based lower bound may be loose due to quantisation noise in the auxiliary channel [4, Fig. 2]. Second, in the small phase noise region, the upper bound may be loose since, for a large number of amplitude values L, the bound on the phase component $I(\Phi; \Psi|\mathcal{A})$ in (6) may not be tight.

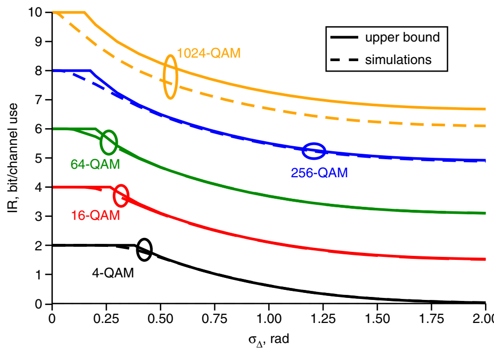

Fig. 2 *IR, as function of σ_Δ, for various QAM modulation schemes at high SNR*

Conclusions and discussion: In this Letter, we derived a simple upper bound on the IR of phase noise-limited communications. Since the IR is a non-decreasing function of the SNR, the bound derived assuming a high-SNR regime is valid for any SNR. The presented upper bound extends previous literature work valid for PSK modulations to general linearly modulated signals, such as QAM or APSK schemes.

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One or more of the Figures in this Letter are available in colour online.

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References

1 Rutman, J., and Walls, F.: 'Characterization of frequency stability in precision frequency sources', *Proc. IEEE*, 1991, **79**, (7), pp. 952–960
2 Huang, K.-C., and Wang, Z.: 'Millimeter wave communication systems' (John Wiley & Sons, Piscataway, NJ, USA, 2011)
3 Björnson, E., Matthaiou, M., Pitarokoilis, A., and Larsson, E.G.: 'Distributed massive MIMO in cellular networks: impact of imperfect hardware and number of oscillators'. Proc. European Signal Processing Conf. (EUSIPCO), Nice, France, August 2015, available on arXiv
4 Barletta, L., Magarini, M., and Spalvieri, A.: 'Estimate of information rates of discrete-time first-order Markov phase noise channels', *IEEE Photonics Technol. Lett.*, 2011, **23**, (21), pp. 1582–1584
5 Goebel, G., Essiambre, R., Kramer, G., Winzer, P.J., and Hanik, N.: 'Calculation of mutual information for partially coherent Gaussian channels with applications to fiber optics', *IEEE Trans. Inf. Theory*, 2011, **57**, (9), pp. 5720–5736
6 Colavolpe, G.: 'Communications over phase noise channels: a tutorial review', *Int. J. Satell. Commun. Netw.*, 2014, **32**, (3), pp. 167–185
7 Martalò, M., Tripodi, C., and Raheli, R.: 'On the information rate of phase noise-limited communications'. Proc. Information Theory and Applications Workshop (ITA), San Diego, CA, USA, February 2013, pp. 1–7
8 Mardia, K., and Jupp, P.E.: 'Directional statistics' (John Wiley & Sons, Chichester, UK, 2000)
9 Arnold, D.M., Loeliger, H.-A., Vontobel, P.O., Kavcic, A., and Zeng, W.: 'Simulation-based computation of information rates for channels with memory', *IEEE Trans. Inf. Theory*, 2006, **52**, (8), pp. 3498–3508